

Approximating Connections in Loop Quantum Gravity

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Abstract

We discuss the action of the configuration operators of loop quantum gravity. In particular, we derive the generalised eigenbasis for the Wilson loop operator and show that the transformation between this basis and the spin-network basis is given by an expansion in terms of Chebyshev polynomials.

These results are used to construct states which approximate connections on the background 3-manifold in an analogous way that the weave states reproduce area and volumes of a given 3-metric. This should be necessary for the construction of genuine semi-classical states that are peaked both in the configuration and momentum variables.

1 Introduction

One of the main challenges facing non-perturbative (loop) quantum gravity to date is to show how general relativity is reproduced in the appropriate classical limit. Semi-classical states that have been considered so far are the weaves [8, 4], which are attempts to approximate classical geometries on a background spatial 3-manifold Σ . More precisely, one requires that given a 3-metric g_{ab} on Σ , expectation values of areas and volumes in the weave state are given by the values determined by g_{ab} .

However, area and volume operators only depend on the momentum variables and we expect that to construct genuine semi-classical or coherent states configuration variables need to be approximated as well. One state corresponding to flat space that satisfies these requirements has been constructed in [1]. To investigate such states in more detail and provide further examples better control of the configuration

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operators is required. In loop quantum gravity these are the cylindrical functions of connections, which essentially determine holonomies on paths embedded in Σ .

In this paper we tackle these issues, and construct states which approximate connections in the same sense that weaves approximate 3-metrics. In particular, we consider the operator operator \mathcal{T}_1 , which is a Wilson loop in the fundamental representation of $SU(2)$. Given any connection A on Σ we show how states can be constructed that give expectation values of \mathcal{T}_1 corresponding to A . We call these states “holonomy weaves”.

To achieve this goal we determine the generalised eigenvectors of the operator \mathcal{T}_1 , which can be understood using the following analogy. It is known that in infinite dimensions operators with continuous spectrum will in general have non-normalisable eigenvectors. This is already evident in elementary quantum mechanics where the operator $\hat{p} = -i\hbar d/dx$ has the improper eigenvectors e^{ipx} . Nevertheless, in practice we can consider physically relevant wave packets or smeared states. The analogue of an expansion of an arbitrary state in terms of the momentum eigenvectors is given by the familiar Fourier transform:

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) e^{ipx} dp,$$

where $f \in \mathcal{L}^2(\mathbb{R})$.

To make the above ideas more rigorous one can introduce the concepts of Gel’fand triples or rigged Hilbert spaces [7]. In this context the functions e^{ipx} are considered as linear functionals acting on a subspace $\mathcal{D} \subset \mathcal{L}^2(\mathbb{R})$, where the action $e^{ipx}[f]$ is just given by the above integral and the subspace \mathcal{D} is determined by the requirement that the integral is well-defined. They are generalised eigenvectors in the sense that:

$$(\hat{p}' e^{ipx})[f] \equiv e^{ipx}[\hat{p}f] = p e^{ipx}[f],$$

for all $f \in \mathcal{D}$, where \hat{p}' denotes the natural dual action of \hat{p} .

The key to this approach is that an analogous construction is always possible for any self-adjoint operator. In fact it can be shown that there is a sense in which every self-adjoint operator has a complete set of generalised eigenvectors. In this paper we apply these results to loop quantum gravity. In particular, we will construct the generalised eigenvectors of the \mathcal{T}_1 operator and show that the analogue of the above Fourier transform is an expansion in term of Chebyshev polynomials. This lets us transform between the spin-network basis and the generalised eigenbasis of the holonomy operator, which as we shall see greatly simplifies our search for the holonomy weave.

2 Generalised eigenvectors of \mathcal{T}_1

2.1 Preliminaries

We begin our discussion by summarising the main facts from loop quantum gravity that will be needed in the following. In a series of papers (e.g. [9, 3, 5]) a rigorous framework for quantising canonical general relativity in the new variables [2, 6] has been developed. In this approach the classical phase space is co-ordinatised by $\mathfrak{su}(2)$ valued connection one-forms A_a^i on a spatial three manifold¹ Σ and a conjugate desitised triad \tilde{E}_i^a , which takes values in the dual of the Lie algebra $\mathfrak{su}(2)$. The spatial index a and the Lie algebra index i will be suppressed in the following.

The quantum theory is given by a Hilbert space \mathcal{H} of cylindrical functions of connections. Cylindrical because they depend on connections only via their holonomies on finite graphs. More precisely, given a set of piecewise analytic paths $\{\gamma_1, \dots, \gamma_n\}$ that form a Graph Γ embedded in Σ we consider the space generated by *gauge invariant*² functions of the type:

$$\Psi_{\Gamma, f}(A) = f(H_{\gamma_1}(A), \dots, H_{\gamma_n}(A)),$$

where $H_{\gamma_i}(A)$ is the holonomy of the connection A along the path γ_i , which takes values in $SU(2)$ and f is a function from $SU(2)^n$ to \mathbb{C} . Completion of this function space in the appropriate norms gives us the Hilbert space \mathcal{H} . It is equipped with the inner product:

$$\langle \Psi_1 | \Psi_2 \rangle = \int_{SU(2)^n} f_1^*(g_1, \dots, g_n) f_2(g_1, \dots, g_n) dg_1 \cdots dg_n. \quad (1)$$

Here we make use of the fact that if the functions f_1 and f_2 have a different number of arguments, say $f_1 : SU(2)^m \rightarrow \mathbb{C}$ with $m < n$, we can trivially extend f_1 to a function on $SU(2)^n$, which does not depend on the last $n - m$ arguments. It can be shown that \mathcal{H} is spanned by the so-called spin-network functions which are generalisations of the Wilson loop. We will be making use of this fact later.

Elementary operators on this space are given by the cylindrical functions, which act multiplicatively and by certain derivations on them. We will be concerned with the configuration operators, which capture information about the connection. In particular, we will study the spectrum of the Wilson loop operator $\mathcal{T}_1 = \text{Tr} [\rho_1(H_\ell(A))]$, where ρ_1 is the fundamental representation of $SU(2)$ and ℓ is a closed loop in Σ . The action of \mathcal{T}_1 is given by:

$$(\mathcal{T}_1 \Psi_{\Gamma, f})(A) = \text{Tr} [\rho_1(H_\ell(A))] f(H_{\gamma_1}(A), \dots, H_{\gamma_n}(A)).$$

In the next section we will be looking for eigenstates of the operator \mathcal{T}_1 . To do this we first restrict our attention to the action of \mathcal{T}_1 in the subspace \mathcal{H}_ℓ of \mathcal{H} given by

¹ Σ is usually taken to be compact. For many applications e.g. asymptotically flat spaces, the non-compact case is more interesting. An extension of the above has been developed in [1].

²Gauge invariance means invariance under $SU(2)$ gauge transformations of the connection, which is required by the constraints of general relativity. We will not be considering the diffeomorphism or Hamiltonian constraints in this paper.

cylindrical function based on ℓ . In the final section we will show how the results obtained can be extended to deal with more general states.

2.2 Chebyshev Polynomials

We begin with the observation that the operator \mathcal{T}_1 has no proper eigenstates, which is to be expected since as a multiplicative operator its spectrum should be the continuous interval $[-2, 2]$. To explore this in more detail we note that any gauge invariant cylindrical function $\Psi(A) = f(H_\ell(A)) \in \mathcal{H}_\ell$ can be expanded in the spin-network basis with coefficients $\psi[p]$:

$$\Psi = \sum_{p=0}^{\infty} \psi[p] \mathcal{T}_p, \quad (2)$$

where $\mathcal{T}_p(A) = \text{Tr} [\rho_p(H_\ell(A))]$ and ρ_p denote the representations of $\text{SU}(2)$ in colour notation, i.e. $\dim(\rho_p) = p + 1$. Eigenvectors Ψ_x of \mathcal{T}_1 have to satisfy:

$$\mathcal{T}_1 \Psi_x = x \Psi_x,$$

where $x \in [-2, 2]$. In colour notation the representation theory of $\text{SU}(2)$ implies that:

$$\mathcal{T}_1(A) \mathcal{T}_p(A) = \mathcal{T}_{p+1}(A) + \mathcal{T}_{p-1}(A), \quad (3)$$

for all connections³ A and $p \geq 1$. Using this equality the eigenvalue equation becomes:

$$\psi_x[1] \mathcal{T}_0 + \sum_{p=1}^{\infty} (\psi_x[p-1] + \psi_x[p+1]) \mathcal{T}_p = \sum_{p=0}^{\infty} x \psi_x[p] \mathcal{T}_p,$$

where the $\psi_x[p]$ are the expansion coefficients of the state Ψ_x . Because of the independence of the \mathcal{T}_p 's this gives us the recursion relation:

$$\begin{aligned} \psi_x[1] - x \psi_x[0] &= 0 \\ \psi_x[p+1] - x \psi_x[p] + \psi_x[p-1] &= 0. \end{aligned}$$

If we set choose⁴ $\psi[0] = 1$ then the above equations define the modified Chebyshev Polynomials $\psi_x[p] = S_p(x)$, which are related to the more usual Chebyshev polynomials of the second kind by $S_p(x) = U_p(x/2)$ i.e.:

$$\begin{aligned} \psi_x[0] &= 1 \\ \psi_x[1] &= x \\ \psi_x[2] &= x^2 - 1 \\ \psi_x[3] &= x^3 - 2x \\ &\vdots \end{aligned}$$

³Equivalently all results in this paper can be seen as dealing with the theory of functions on $\text{SU}(2)$.

⁴ This freedom is equivalent to a choice of norm.

This expansion of Ψ_x in terms of the $\psi[p]$ does not converge for all $x \in [-2, 2]$ and hence there are in general no proper eigenstates of the \mathcal{T}_1 operator. As explained in the introduction, one way of approaching this problem is via the Gel'fand triple construction and the use of generalised eigenvectors⁵. To do this we look for linear functionals F_x on some dense subspace $\mathcal{D} \subset \mathcal{H}_\ell$ that satisfy:

$$(\mathcal{T}'_1 F_x)[\Psi] \equiv F_x[\mathcal{T}_1 \Psi] = x F_x[\Psi],$$

for all $\Psi \in \mathcal{D}$, where the subset \mathcal{D} will be determined more precisely later. Using the basis expansion (2) of Ψ and the linearity of F_x we obtain:

$$\psi[0]F_x[\mathcal{T}_1] + \sum_{p=1}^{\infty} \psi[p] (F_x[\mathcal{T}_{p-1}] + F_x[\mathcal{T}_{p+1}]) = \sum_{p=0}^{\infty} x\psi[p]F_x[\mathcal{T}_p]$$

Again, because the coefficients $\psi[p]$ are arbitrary this is solved by the Chebyshev polynomials:

$$F_x[\mathcal{T}_p] = S_p(x).$$

The difference is that now the polynomials $S_p(x)$ define a genuine basis in the space of functionals on \mathcal{D} . Since the cylindrical function operators all commute amongst each other we expect that the generalised eigenvectors F_x of \mathcal{T}_1 will also diagonalise the operators \mathcal{T}_p , for all p . This is indeed the case as we show in Appendix A.

In the next section we will discuss how these generalised eigenvectors are related to the spin-network basis.

3 Transformation of bases

The usefulness of generalised eigenvectors stems from the fact that they allow us to expand states in \mathcal{H}_ℓ in a very intuitive fashion. Moreover there is a sense in which \mathcal{H}_ℓ is spanned by the generalised vectors. We now provide the details of the transformation between the spin-network basis and the generalised eigenbasis. As we demonstrate later this greatly simplifies calculations that are otherwise very intractable.

Let us define the function $\tilde{\psi}(x)$ on the closed interval $[-2, 2]$:

$$\tilde{\psi}(x) \equiv w(x)^{\frac{1}{2}} F_x[\Psi] = w(x)^{\frac{1}{2}} \sum_{p=0}^{\infty} \psi[p] S_p(x), \quad (4)$$

where $w(x)$ is the weight function:

$$w(x) = \frac{1}{2\pi} \sqrt{4 - x^2}$$

⁵Since \mathcal{T}_1 acts multiplicatively one might expect the eigenvectors to be delta function distributions either on the space of connections or the group $SU(2)$. This is indeed correct and there is a close relation between these delta functions and the generalised eigenvectors we are about to construct as will be explored in future work.

As the notation implies $\tilde{\psi}(x)$ will be the expansion coefficients of the state Ψ in the generalised eigenbasis of the operator \mathcal{T}_1 . In particular, we will see that $\tilde{\psi}(x)^2$ gives the probability of obtaining the value x when making a measurement of the trace of the holonomy around the loop ℓ . Note that for the definition of $\tilde{\psi}(x)$ we need to require that the series in equation (4) converges for all $x \in [-2, 2]$. The set of all $\Psi \in \mathcal{H}_\ell$ for which this is true will be denoted by \mathcal{P} . We will come back to this point later.

First let us investigate the inverse of the above transformation, which uses the orthogonality properties of the Chebyshev polynomials. In particular we have:

$$\int_{-2}^2 w(x) S_n(x) S_m(x) dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Using this we find:

$$\int_{-2}^2 w(x)^{\frac{1}{2}} \tilde{\psi}(x) S_p(x) dx = \int_{-2}^2 w(x) \sum_{k=0}^{\infty} \psi[k] S_k(x) S_p(x) dx = \psi[p]. \quad (5)$$

This represents the desired expansion of Ψ in terms of generalised eigenvectors of \mathcal{T}_1 with eigenvalues in the interval $[-2, 2]$. Note that in the last step we had to exchange the order of integration and taking the limit, which is valid if $\sum_{p=0}^{\infty} \psi[p] S_p(x)$ converges *uniformly* on $[-2, 2]$. Let \mathcal{U} be the set of Ψ with such coefficients. Even if $\Psi \notin \mathcal{U}$ it might still be true that the series in equation (4) with the coefficients given by equation (5) converges pointwise to the function $\tilde{\psi}(x)$, i.e. that $\tilde{\psi}(x)$ has a convergent Chebyshev expansion. The functions Ψ with this property will be the set \mathcal{D} that we need in our Gel'fand triple construction. In general, we have $\mathcal{U} \subset \mathcal{D} \subset \mathcal{P}$, and the precise determination of \mathcal{D} is to our knowledge not yet available. In practice, we will have to make sure that functions we use are of the correct type by checking if the expansions (4) and (5) are compatible.

There is a general theorem that states that to any operator there is a complete set of generalised eigenvectors, which means that given any state there is unique expansion in terms of them. To see this in the present context we express the norm of Ψ in terms of the coefficients $\tilde{\psi}(x)$:

$$\begin{aligned} \sum_{p=0}^{\infty} \psi[p]^2 &= \sum_{p=0}^{\infty} \psi[p] \int_{-2}^2 w(x)^{\frac{1}{2}} \tilde{\psi}(x) S_p(x) dx \\ &= \int_{-2}^2 w(x)^{\frac{1}{2}} \tilde{\psi}(x) \sum_{p=0}^{\infty} \psi[p] S_p(x) dx \\ &= \int_{-2}^2 \tilde{\psi}(x)^2 dx, \end{aligned}$$

which is Parseval's equation for orthogonal polynomials. In particular, we deduce that if $\tilde{\psi}(x) = \tilde{\psi}'(x)$ for all $x \in [-2, 2]$, then these functions will determine the same spin-network coefficients $\psi[p]$, i.e. $\psi[p] = \psi'[p]$ for all p .

The main practical benefit of the generalised eigenvectors comes through the natural expression of operator actions. We show that for the expectation value of any operator \hat{A} we have:

$$\langle \Psi | \hat{A} \Psi \rangle = \int_{-2}^2 w(x)^{\frac{1}{2}} \tilde{\psi}(x) F_x[\hat{A} \Psi] dx,$$

where the F_x are the generalised eigenvectors of \mathcal{T}_1 as before. The above follows from equations (5) and (4) since:

$$\begin{aligned} \langle \Psi | \hat{A} \Psi \rangle &= \sum_{p=0}^{\infty} \psi[p] (\hat{A} \Psi)[p] \\ &= \sum_{p=0}^{\infty} \psi[p] \int_{-2}^2 w(x) F_x[\hat{A} \Psi] S_p(x) dx \\ &= \int_{-2}^2 w(x)^{\frac{1}{2}} \tilde{\psi}(x) F_x[\hat{A} \Psi] dx, \end{aligned}$$

where $(\hat{A} \Psi)[p] = \langle \mathcal{T}_p | \hat{A} \Psi \rangle$ are the expansion coefficients of $\hat{A} \Psi$ in the spin-network basis.

As an application we show how the expectation value of the operator \mathcal{T}_1 takes on an intuitive form in the generalised basis, which will be used in the next section. Using the defining equation of the generalised eigenvectors, $F_x[\mathcal{T}_1 \Psi] = x F_x[\Psi]$, we derive:

$$\begin{aligned} \langle \Psi | \mathcal{T}_1 \Psi \rangle &= \int_{-2}^2 w(x)^{\frac{1}{2}} \tilde{\psi}(x) \sum_{p=0}^{\infty} \psi[p] F_x[\mathcal{T}_1 \Psi] dx \\ &= \int_{-2}^2 x \tilde{\psi}(x) w(x)^{\frac{1}{2}} \sum_{p=1}^{\infty} S_p(x) dx \\ &= \int_{-2}^2 x \tilde{\psi}(x)^2 dx, \end{aligned}$$

which shows that $\tilde{\psi}(x)$ can be interpreted as a probability amplitude.

4 Holonomy weaves

We make use of the results in the previous section to construct a state for loop quantum gravity that approximates a given connection on the spatial manifold Σ . This is to be seen in analogy to previous constructions of weaves which approximate areas and volumes on Σ given a background 3-metric.

As in the weave construction our state is based on a background graph embedded in Σ , field excitations will be concentrated on the edges of the graph. For simplicity

we consider a graph Γ , which is just a union of loops ℓ_i embedded in Σ . This union is finite if Σ is compact. Otherwise, we need to use a modification of the standard approach to loop gravity such as the one given in [1] to make states well-defined. For ease of exposition we may restrict ourselves to the compact case in the following. At the end of this section we will briefly discuss the implications of the choice of graph and what modifications are possible.

The holonomy weave, which we will denote by \mathcal{W} , is the product of normalised cylindrical functions $\Psi_i(A) = f_i(H_{\ell_i}(A))$ based on the loop ℓ_i :

$$\mathcal{W}(A) = \prod_{i=0}^n \Psi_i(A),$$

where the product ranges over all loops in Γ . This state is characterised by the requirement that the expectation value of $\mathcal{T}_1^i(A) \equiv \text{Tr}[H(\ell_i, A)]$ for any loop ℓ_i in Γ is given by the value we expect given the background connection A that we wish to approximate, i.e.:

$$\langle \mathcal{W} | \mathcal{T}_1^i | \mathcal{W} \rangle = \text{Tr}[\rho_1(H_{\ell_i}(A))],$$

for all ℓ_i .

The link to the results of the previous sections comes because the definition of the inner product (1) implies that the above expectation value depends only on the state Ψ_i in \mathcal{W} . More precisely we have:

$$\begin{aligned} \langle \mathcal{W} | \mathcal{T}_1^i | \mathcal{W} \rangle &= \int_{\text{SU}(2)^n} \prod_{j=0}^n f_j^*(g_j) \mathcal{T}_1^i(g_i) \prod_{k=0}^n f_k(g_k) dg_0 \dots dg_n \\ &= \prod_{j=0}^n \int_{\text{SU}(2)^n} f_j^*(g_j) \mathcal{T}_1^i(g_i) f_j(g_j) dg_j \\ &= \langle \Psi_i | \mathcal{T}_1^i | \Psi_i \rangle, \end{aligned}$$

since all Ψ_j are normalised.

Hence, to reproduce the holonomies of any connection, to the accuracy that the graph allows, we need to choose the Ψ_j 's in such a way that they have the desired expectation value of \mathcal{T}_1 . Since we can do this independently for each of the Ψ_j and since these functions are each based on just one loop we can make use of the results of the previous section. To appreciate how our task has simplified let us first see what this problem looks like in the standard spin-network basis. Using equation (3) we deduce:

$$\langle \Psi | \mathcal{T}_1 | \Psi \rangle = 2 \sum_{p=0}^{\infty} \psi[p] \psi[p+1].$$

Choosing coefficients $\psi[p]$ to reproduce any of the possible expectation values seems intractable. However, the formula derived in the last section allows us to write:

$$\langle \Psi | \mathcal{T}_1 | \Psi \rangle = \int_{-2}^2 x \tilde{\psi}(x)^2 dx.$$

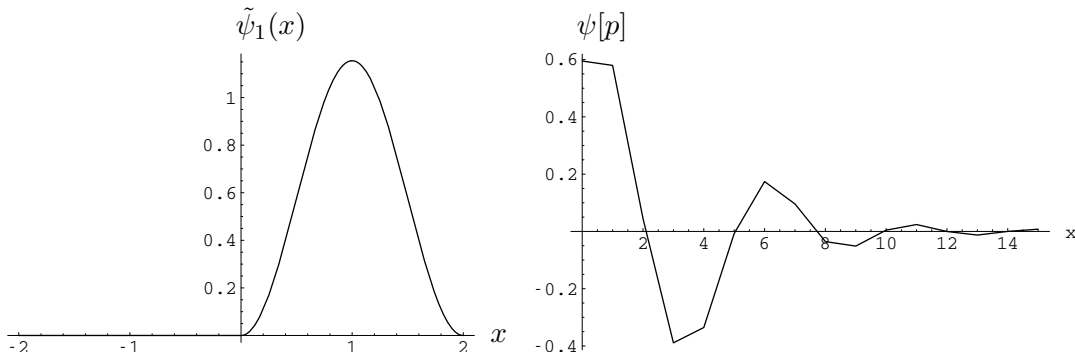


Figure 1: $\tilde{\psi}_1(x)$ and the normalised spin-network coefficients $\psi[p]$ for the state Ψ_1 in the range $p \in [0, 15]$.

Hence, to obtain an expectation value $a \in]-2, 2[$ of \mathcal{T}_1 we need to choose a function $\tilde{\psi}_a(x)$ on $[-2, 2]$ such that:

$$\int_{-2}^2 x \tilde{\psi}_a(x)^2 dx = a.$$

This is solved by any function $\tilde{\psi}_a(x)$ that is symmetric about a and is normalised on the interval $[-2, 2]$. For such a function $\tilde{\psi}_a(x)$ to define a state $\Psi_a \in \mathcal{H}_\ell$ we need the further requirements that $\tilde{\psi}_a(x)$ has a convergent Chebyshev expansion (so that $\Psi_a \in \mathcal{D}$) and also — because of $w(x)^{\frac{1}{2}}$ in equation (4) — that $\tilde{\psi}_a(-2) = \tilde{\psi}_a(2) = 0$. Note that it is impossible to construct a state with the expectation value 2 or -2 . This is to be expected since states with expectation values corresponding to the boundary of a closed spectrum are necessarily eigenstates, but as we have seen these are not normalisable. Nevertheless, it is possible to obtain expectation values arbitrarily close to the boundary values. This raises interesting questions of whether flat space can be constructed as a cylindrical function state.

As an example let us construct a normalised function peaked around $a = 1$ to obtain a state Ψ_1 with expectation value 1. We define:

$$\tilde{\psi}_1(x) = \begin{cases} 0 & x \in [-2, 0] \\ \frac{1}{\sqrt{3}}(\cos[\pi(x-1)] + 1) & x \in [0, 2] \end{cases}$$

$\tilde{\psi}_1(x)$ and the corresponding expansion coefficients $\psi[p]$ in the spin-network basis are shown in figure 1. The $\psi[p]$ fall off rapidly enough to make the transformations between both bases well-defined.

If weaves are truly to represent semi-classical states then the requirement that appropriate expectation values are reproduced is not sufficient. In addition restrictions have to be made on the standard deviations from the average so that if measurements of areas, volumes, or holonomies are made on scales large compared to the Planck scale we do not obtain any deviations from classical values (c.f. [1]). In the context of the holonomy weaves we note that we can make deviations around the

expectation value of the \mathcal{T}_1 operator arbitrarily small by sharpening the peak of the functions $\tilde{\psi}_a(x)$.

This is adequate if we happen to measure the holonomy around a loop that is precisely one of the ℓ_i included in \mathcal{W} . Since this is almost never the case we need further requirements on the nature of the graph underlying the definition of the holonomy weave. Since excitations are concentrated on the edges of the graph we would ideally like to cover as much as the manifold Σ as possible to approximate a smooth connection field. But to obtain physically viable states we have to make sure that these states are weaves in the geometric sense as well. This places restrictions on density of vertices and sizes of loops as determined in [8]. In future work we would like to combine the results from the geometric and the holonomy weaves in order to construct genuine coherent states.

Acknowledgements

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Appendix A

We show that generalised eigenvectors F_x of \mathcal{T}_1 are also generalised eigenvectors of \mathcal{T}_p with the correct eigenvalues, i.e.:

$$F_a[\mathcal{T}_p\Psi] = bF_a[\mathcal{T}_p\Psi],$$

for all $\Psi \in \mathcal{H}_\ell$, where $a = \mathcal{T}_1(A)$ and $b = \mathcal{T}_p(A)$ for some connection A .

First we note the useful relation:

$$\mathcal{T}_p(A) = S_p(\mathcal{T}_1(A)). \quad (6)$$

This follows from the defining recursion relations for the polynomials, $S_{p+1}(x) - xS_p(x) + S_{p-1}(x) = 0$, and equation (3) by setting $x = \mathcal{T}_1(A)$.

Next we show that for $p \leq n$:

$$r_p(x) \equiv \sum_{l=0}^p S_{n-p+2l}(x) = S_p(x)S_n(x)$$

Consider $r_{p+2} + r_p$, where we suppress the x dependence for notational convenience:

$$\sum_{l=0}^{p+2} S_{n-(p+2)+2l} + r_p = 2r_p + S_{n-(p+2)} + S_{n+p+2}$$

Using $S_p = xS_{p+1} - S_{p+2}$ we get:

$$\begin{aligned} r_{p+2} + r_p &= x(S_{n-(p+1)} + S_{n+p+1}) + S_{n-p} + S_{n+p} + 2 \sum_{l=1}^{p-1} S_{n-p+2l} \\ &= x(S_{n-(p+1)} + S_{n+p+1}) + r_p + r_{p-2} \end{aligned} \quad (7)$$

Repeating these steps we arrive at:

$$r_{p+2} + r_p = x(S_{n-(p+1)} + \dots + S_{n-3} + S_{n+3} + \dots + S_{n+p+1}) + r_2 + r_0,$$

for even p , and

$$r_{p+2} + r_p = x(S_{n-(p+1)} + \dots + S_{n-4} + S_{n+4} + \dots + S_{n+p+1}) + r_3 + r_1,$$

for odd p . Now:

$$r_2 + r_0 = S_{n-2} + S_{n+2} + 2S_n = x(S_{n-1} + S_{n+1}),$$

using the defining recursion relation for S , and:

$$r_3 + r_1 = x(S_{n-2} + S_{n+2}) - S_{n-1} - S_{n+1} + 2S_{n+1} = x(S_{n-2} + S_n + S_{n+2}),$$

using equation (7). Hence in both cases we have:

$$r_{p+2} + r_p = x \sum_{l=0}^{p+1} S_{n-(p+1)+2l} = xr_{p+1}.$$

Furthermore we have the initial conditions $r_0 = S_n$ and $r_1 = S_{n-1} + S_{n+1} = xS_n$. Together with the above this defines the modified Chebyshev polynomials and:

$$r_p = S_p S_n,$$

which is the desired result.

Now it is easy to show:

$$F_a[\mathcal{T}_p \mathcal{T}_n] = bF_a[\mathcal{T}_n],$$

for all n . Where $a = \mathcal{T}_1(A)$ is the eigenvalue of the operator \mathcal{T}_1 and $b = S_p(a)$ is the value of $\mathcal{T}_p(A)$ according to equation (6). Indeed if $p \leq n$:

$$\begin{aligned} F_a[\mathcal{T}_p \mathcal{T}_n] &= \sum_{l=0}^p F_a[\mathcal{T}_{n-p+2l}] \\ &= \sum_{l=0}^p S_{n-p+2l}(a) \\ &= S_p(a) S_n(a) \\ &= bF_a[\mathcal{T}_n] \end{aligned}$$

If $p > n$ we get the same result by symmetry: $\mathcal{T}_p \mathcal{T}_n = \mathcal{T}_n \mathcal{T}_p$. Hence the the eigenvalue equation is satisfied for all \mathcal{T}_n and consequently for all $\Psi \in \mathcal{H}_\ell$ since they are spanned by the \mathcal{T}_n \square .

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